Part 3: Coding for DNA-Based Data Storage

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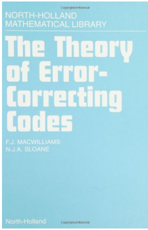

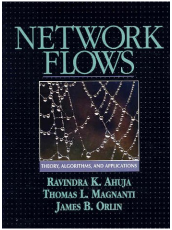

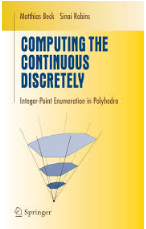

Coding Problems

Fundamentally New Coding Questions

- DNA Profile and Uniquely Reconstructable Codes (IDT Synthesis, Illumina Sequencing)

  Codes for DNA Sequence Profiles, IT, 2016; Unique Reconstruction of Coded Strings from Multiset Substring Spectra, ISIT 2018.

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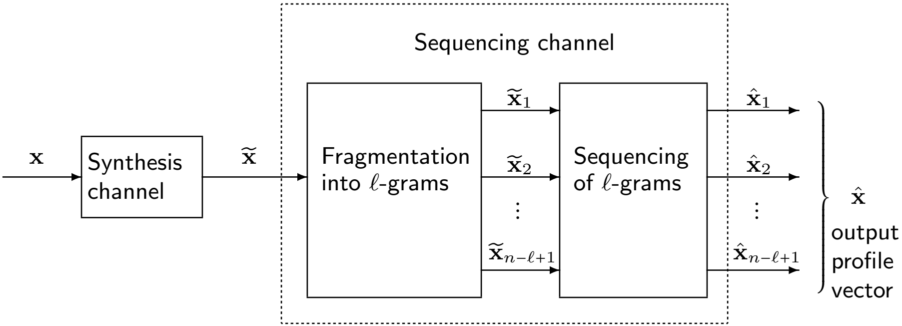
- **Codes in the Damerau Distance (Aging)**

  Codes in the Damerau Distance for Deletion and Adjacent Transposition Correction, IT 2018.

DNA Profile Codes

DNA Storage Channel



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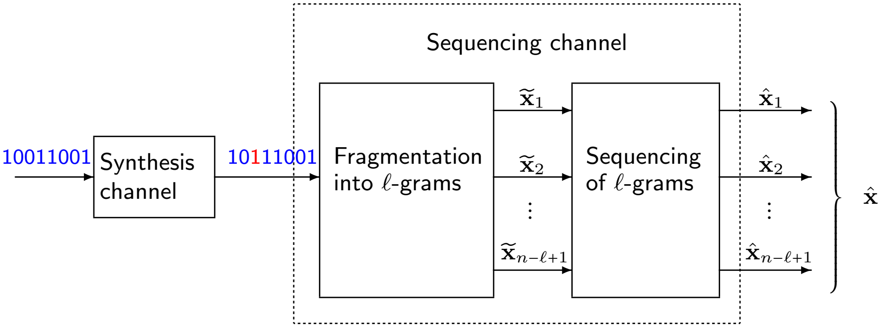

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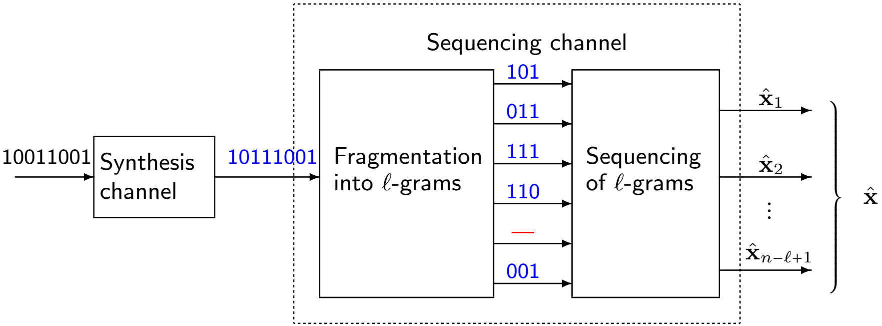

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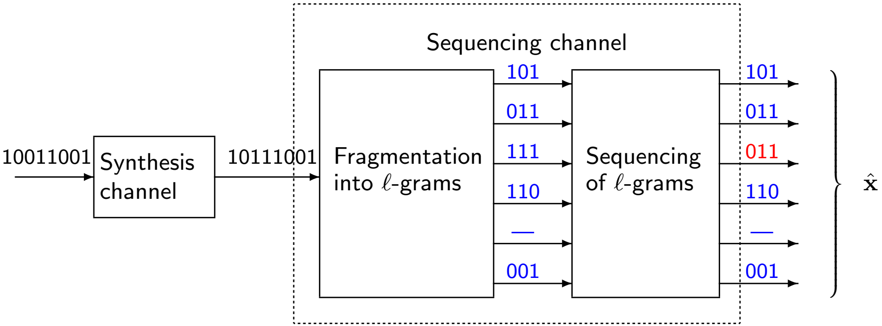
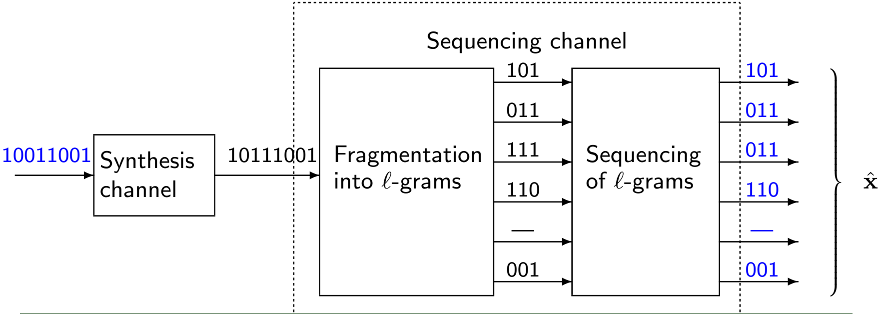


DNA Storage Channel – Output Profile Vectors



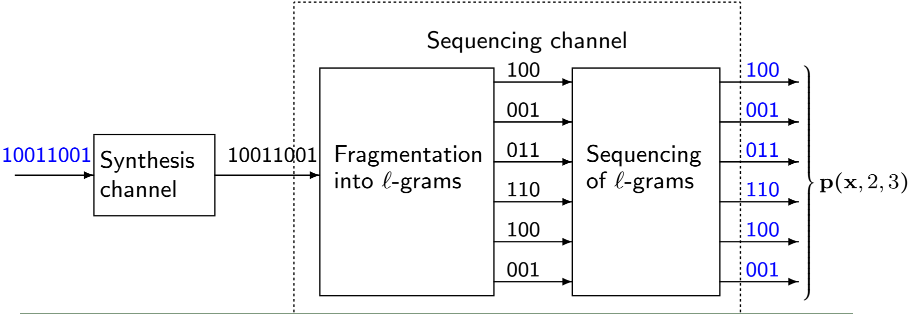
Output profile vector

Given an input sequence 10011001, we obtain an output profile vector that reflects the (possibly erroneous) count of each substring

$$\begin{array}{cccccccc} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\ (0, & 1, & 0, & 2, & 0, & 1, & 1, & 0). \end{array}$$

Note: position of substring is not known!
Note: input is a binary sequence, while the output is an integer-valued vector.

DNA Storage Channel – Profile Vectors



Profile vector

Given an input sequence $\mathbf{x}$ = 10011001, its profile vector denoted by $\mathbf{p}(\mathbf{x}, q, \ell)$ reflects the actual count of each substring

$$\begin{array}{cccccccc} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\ (0, & 2, & 0, & 1, & 2, & 0, & 1, & 0). \end{array}$$

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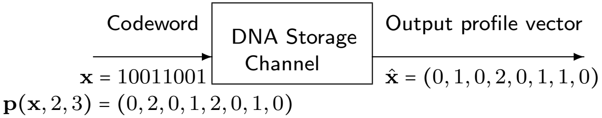

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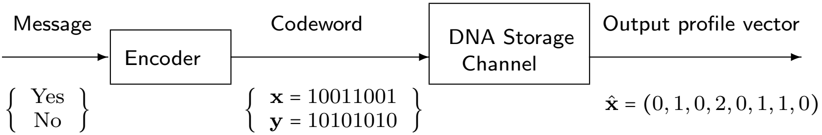
Code Design Criteria



$$
\begin{array}{lcccccccccl}
 & & 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 & \\
\mathbf{p}(\mathbf{x};2,3) & = & (0, & 2, & 0, & 1, & 2, & 0, & 1, & 0) & \leftarrow \text{dist with } \hat{\mathbf{x}} \text{ is } 3 \\
\mathbf{p}(\mathbf{y};2,3) & = & (0, & 0, & 3, & 0, & 0, & 3, & 0, & 0) & \leftarrow \text{dist with } \hat{\mathbf{x}} \text{ is } 5
\end{array}
$$

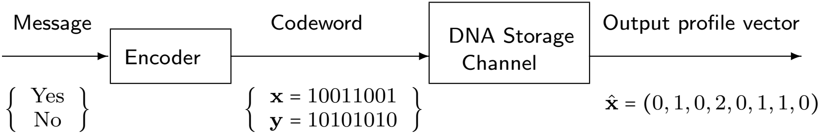
Criterion 1

Codewords whose profile vectors are far from each other.

We define the ℓ-gram distance between $\mathbf{x}$ and $\mathbf{y}$ to be the asymmetric distance between the profile vectors of $\mathbf{x}$ and $\mathbf{y}$.

Define the asymmetric distance as $\max(\Delta(\mathbf{u},\mathbf{v}),\Delta(\mathbf{v},\mathbf{u}))$, where $\Delta(\mathbf{u},\mathbf{v}) = \sum_i \max(u_i - v_i, 0)$.

Code Design Criteria



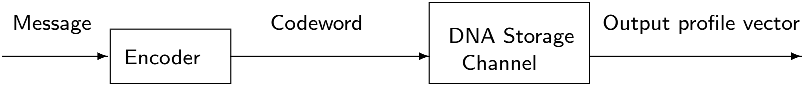
$$
\begin{array}{rcrrrrrrrr}
 & & 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\
\mathbf{p}(\mathbf{x}; 2, 3) & = & (0, & 2, & 0, & 1, & 2, & 0, & 1, & 0) \\
\mathbf{p}(\mathbf{y}; 2, 3) & = & (0, & 0, & 3, & 0, & 0, & 3, & 0, & 0)
\end{array}
$$

Criterion 2

Codewords whose ℓ-substrings are resilient to errors.

Certain reliability considerations in DNA storage sequence designs:

- Balanced profiles of ℓ-substrings. Number of C, G bases needs to be roughly fifty percent.
- Forbidden ℓ-substrings. Certain substrings like GCG and CGC or GGG are more likely to cause sequencing errors.



$$\left\{ \begin{array}{c} \text{Yes} \\ \text{No} \end{array} \right\} \qquad \left\{ \begin{array}{l} \mathbf{x} = 10011001 \\ \mathbf{y} = 10101010 \end{array} \right\} \qquad \hat{\mathbf{x}} = (0,1,0,2,0,1,1,0)$$

$$\begin{array}{rcllllllll} & & 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\ \mathbf{p}(\mathbf{x};2,3) & = & (0, & 2, & 0, & 1, & 2, & 0, & 1, & 0) \\ \mathbf{p}(\mathbf{y};2,3) & = & (0, & 0, & 3, & 0, & 0, & 3, & 0, & 0) \end{array}$$

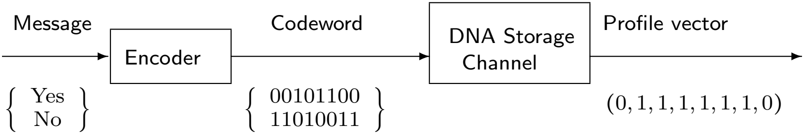
Criterion 2

Codewords whose ℓ-substrings are resilient to errors.

Here, we assume that the ℓ-substrings belong to
$S = \{001, 010, 011, 100, 101, 110\}$.

Fundamental Question - Number of Distinct Profile Vectors

The following example is bad, because the codewords share the same profile vector.



Distinct ℓ-gram Profile Vectors

Define $\mathcal{Q}(n;S)$ to be the set of q-ary words of length n with distinct ℓ-gram profile vectors whose ℓ-grams belong to S.
Determine the size of $\mathcal{Q}(n;S)$.

- n : length of codewords
- q : alphabet size
- ℓ : length of substrings / grams
- S : set of "constrained" substrings (note S is a set of q-ary strings of length ℓ)

Fundamental Question - Code Construction

ℓ-gram Reconstruction Code (GRC)

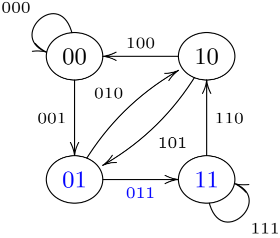
$\mathcal{C} \subseteq \mathcal{Q}(n;S)$ is an $(n,d;S)$-ℓ-GRC if the ℓ-gram distance between any pair of distinct words is at least d.
Construct good $(n,d;S)$-ℓ-GRC. Good means "more codewords."

- n : length of codewords
- q : alphabet size
- ℓ : length of substrings / grams
- S : set of "constraint" substrings (note S is a set of q-ary strings of length ℓ)
- d : minimum ℓ-gram distance between any pair of codewords

Enumeration of Profile Vectors

De Bruijn Graphs

Example: $q = 2$, $\ell = 3$.



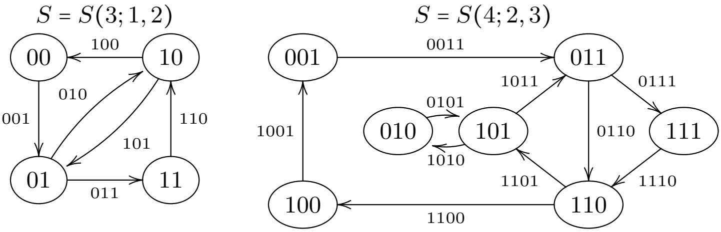
De Bruijn Graphs (de Bruijn, 1946)

Nodes are q-ary strings of length $\ell - 1$.
$(\mathbf{v}, \mathbf{v}')$ is an arc if

$$\begin{array}{ccccc} v_2 & v_3 & & v_{\ell-1} & \\ \shortparallel & \shortparallel & \cdots & \shortparallel & . \\ v'_1 & v'_2 & & v'_{\ell-2} & \end{array}$$

Restricted De Bruijn Graphs

Let $S(\ell; w_1, w_2)$ denote the binary strings of length ℓ with weight between w_1 and w_2.

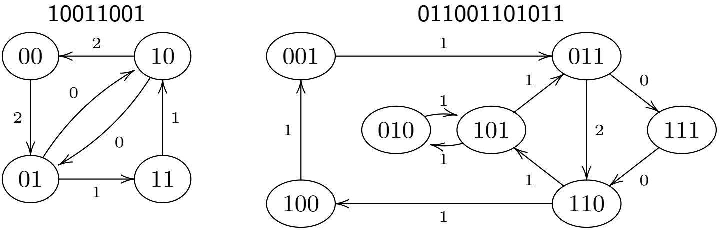


Restricted de Bruijn Graphs $D(S)$ (Ruskey, Sawada, Williams, 2012)

Nodes V are $\ell - 1$-prefixes and -suffixes of strings in S.
$(\mathbf{v}, \mathbf{v}')$ is an arc if

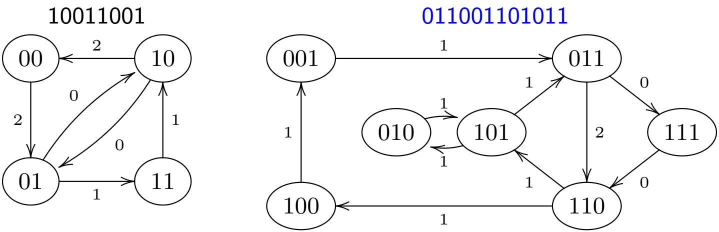
$$\begin{array}{cccc} v_2 & v_3 & & v_{\ell-1} \\ \shortparallel & \shortparallel & \dots & \shortparallel \\ v'_1 & v'_2 & & v'_{\ell-2} \end{array} \quad \text{and} \quad v_1 v_2 \cdots v_{\ell-1} v'_{\ell-1} \in S.$$

Profile Vectors and Flow Vectors



Representing profile vectors of words in $\mathcal{Q}(n;S)$ using the digraph $D(S)$.

Profile Vectors and Flow Vectors



Profile vectors of closed words in $\mathcal{Q}(n;S)$ are flow vectors in $D(S)$.

Closed Words

Closed words that are words that start and end with the same $(\ell - 1)$-gram. Denote the set of q-ary words of length n with distinct ℓ-gram profile whose ℓ-grams belong to S by $\overline{\mathcal{Q}}(n;S)$.

Flow Vectors

Incoming flow is equal to outgoing flow at each node.

Necessary Conditions

Let $\mathbf{u}$ be a profile vector of a closed word. Then $\mathbf{u}$ satisfies the following conditions.

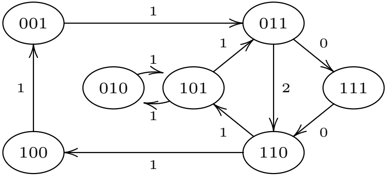
Flow conservations equations:

$$\mathbf{B}\mathbf{u} = \mathbf{0},$$

where $\mathbf{B}$ be the incidence matrix of $D(S)$.

Sum of flows:

$$\mathbf{1}\mathbf{u} = n - \ell + 1.$$



Let $\mathbf{A} = \begin{pmatrix} \mathbf{1} \\ \mathbf{B} \end{pmatrix}$ and $\mathbf{b} = (1, 0, \ldots, 0)^T$. We rewrite the equations as

Necessity

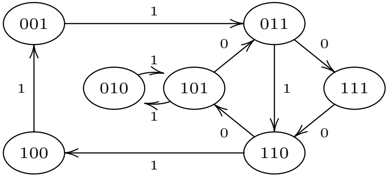
$$\mathbf{A}\mathbf{u} = (n - \ell + 1)\mathbf{b} \text{ and } \mathbf{u} \geq \mathbf{0}.$$

Flow vectors are not always profile vectors

Let $\mathbf{u} \geq \mathbf{0}$ be such that

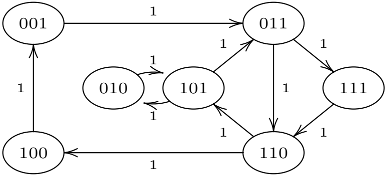
$$\mathbf{A}\mathbf{u} = (n - \ell + 1)\mathbf{b}.$$

This does not imply that $\mathbf{u}$ is a profile vector!



Sufficient Conditions

If all flows are positive, then the flow vector is indeed a profile vector.



Profile vector of 0110101110011.

Sufficiency

$$\mathbf{A}\mathbf{u} = (n - \ell + 1)\mathbf{b} \text{ and } \mathbf{u} > \mathbf{0}.$$

Profile Vectors and Lattice Points

Consider the following two sets of lattice points:

$$\mathcal{F}(n;S) \triangleq \{\mathbf{u} \in \mathbb{Z}^{|S|} : \mathbf{A}\mathbf{u} = (n-\ell+1)\mathbf{b},\ \mathbf{u} \geq \mathbf{0}\},$$

$$\mathcal{E}(n;S) \triangleq \{\mathbf{u} \in \mathbb{Z}^{|S|} : \mathbf{A}\mathbf{u} = (n-\ell+1)\mathbf{b},\ \mathbf{u} > \mathbf{0}\}.$$

$$|\mathcal{E}(n;S)| \leq |\overline{\mathcal{Q}}(n;S)| \leq |\mathcal{F}(n;S)|.$$

Profile Vectors and Lattice Points

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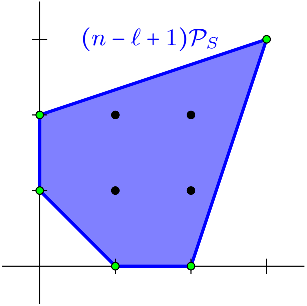
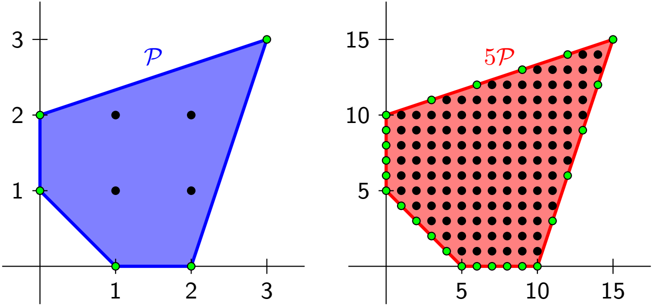
$$|\mathcal{E}(n;S)| \leq |\overline{\mathcal{Q}}(n;S)| \leq |\mathcal{F}(n;S)|.$$

Observations

- Define the polytope
$$\mathcal{P}_S = \{\mathbf{u} \in \mathbb{R}^{|S|} : \mathbf{A}\mathbf{u} = \mathbf{b},\ \mathbf{u} \geq \mathbf{0}\}.$$
- $\mathcal{F}(n;S)$ is the set of lattice points in $(n-\ell+1)\mathcal{P}_S$.
- $\mathcal{E}(n;S)$ is the set of lattice points in the interior of $(n-\ell+1)\mathcal{P}_S$.





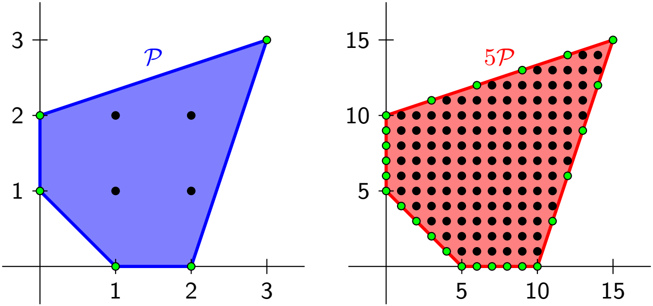
For a polytope $\mathcal{P} \subset \mathbb{R}^N$ and $t \in \mathbb{R}$, the dilation $t\mathcal{P}$ is given by

$$t\mathcal{P} = \{tx : x \in \mathcal{P}\}.$$

The lattice point enumerator for $\mathcal{P}$ is $\mathcal{L}_{\mathcal{P}} : \mathbb{R} \to \mathbb{Z}$ defined by

$$\mathcal{L}_{\mathcal{P}}(t) = |t\mathcal{P} \cap \mathbb{Z}^N|.$$

Lattice Point Enumeration in Dilated Polytopes



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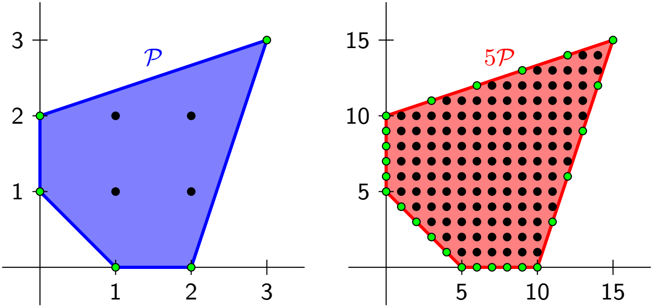
Theorem (Ehrhart)

If $\mathcal{P}$ is a rational D-dimensional polytope, then $\mathcal{L}_\mathcal{P}(t)$ is a "quasipolynomial" in t with degree D.

Lattice Point Enumeration in Dilated Polytopes



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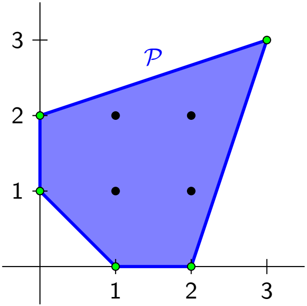
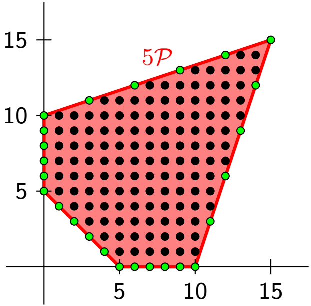
Theorem (Ehrhart-Macdonald's reciprocity)

The number of lattice points in the interior of $t\mathcal{P}$ is given by $(-1)^D \mathcal{L}_{\mathcal{P}}(-t)$, and is thus a "quasipolynomial" with degree D.





For a polytope $\mathcal{P} \subset \mathbb{R}^N$ and $t \in \mathbb{R}$, the dilation $t\mathcal{P}$ is given by

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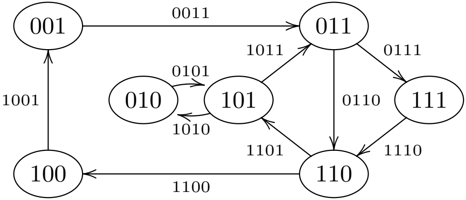
Lemma

The polytope $\mathcal{P}_S$ has dimension $|S| - |V|$ if $D(S)$ is strongly connected.

Main Enumeration Result



Here, $|S| = 10$, $|V| = 7$ and so, the number of distinct profile vectors of closed words is

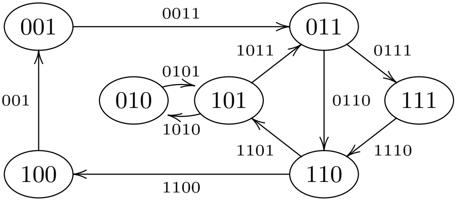
$$|\overline{\mathcal{Q}}(n;S)| = \Theta'(n^3).$$

Theorem

Suppose $D(S)$ is strongly connected. Then $|\mathcal{E}(n;S)|$ and $|\mathcal{F}(n;S)|$ are both "quasipolynomials" in n of the same degree $|S| - |V|$. In particular, $|\overline{\mathcal{Q}}(n;S)| = \Theta'\left(n^{|S|-|V|}\right)$.

- A *quasipolynomial* f is a function in n of the form $c_D(n)n^D + c_{D-1}(n)n^{D-1} + \cdots + c_0(n)$, where $c_D, c_{D-1}, \ldots, c_0$ are periodic functions in n. If c_D is not identically equal to zero, f is said to be of *degree* D.
- $f(n) = \Omega'(g(n))$ means that for a fixed value of ℓ, there exists an integer λ and a positive constant c so that $f(n) \geq cg(n)$ for sufficiently large n with $\lambda|(n - \ell + 1)$. Furthermore, $f(n) = \Theta'(g(n))$ if $f(n) = O(g(n))$ and $f(n) = \Omega'(g(n))$.

Main Enumeration Result



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Results hold if n satisfies certain periodicity conditions.

Code Constructions

Varshamov Codes

Fix d and let p be a prime such that $p > d$ and $p > N$. Choose N distinct nonzero elements $\alpha_1, \alpha_2, \ldots, \alpha_N$ in $\mathbb{Z}/p\mathbb{Z}$ and consider the matrix

$$\mathbf{H} \triangleq \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_N \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^d & \alpha_2^d & \cdots & \alpha_N^d \end{pmatrix}.$$

Pick any vector $\boldsymbol{\beta} \in (\mathbb{Z}/p\mathbb{Z})^N$ and define the code

$$\mathcal{C}(\mathbf{H}, \boldsymbol{\beta}) \triangleq \{\mathbf{u} \in \mathbb{Z}^N : \mathbf{H}\mathbf{u} \equiv \boldsymbol{\beta} \bmod p\}.$$

Theorem (Varshamov, 1973)

$\mathcal{C}(\mathbf{H}, \boldsymbol{\beta})$ *is a code of length* N *with minimum asymmetric distance* $d + 1$.

GRC as Subcodes of Varshamov Codes

Construction I

Let $\mathbf{p}\mathcal{Q}(n;S)$ be the set of distinct profile vectors of words in S and $N = |S|$. Then $\mathcal{C}(\mathbf{H},\boldsymbol{\beta}) \cap \mathbf{p}\mathcal{Q}(n;S)$ is an $(n, d+1; S)$-ℓ-gram reconstruction code.

For example, let $q = 2$, $\ell = 3$, $S = \{001, 010, 011, 100, 101, 110\}$ and so, $N = 6$. Let $d = 3$ and we pick $p = 7$,

$$\mathbf{H} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 2 & 4 & 1 \end{pmatrix} \text{ and } \boldsymbol{\beta} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then $\mathcal{C}(\mathbf{H},\boldsymbol{\beta})$ contains the following words:

$$\begin{array}{llllll}
(4,0,0,1,0,1) & & & (0,1,1,4,0,0) & & \\
(2,2,0,2,0,0) & \leftrightarrow & \texttt{00100100} & (0,1,0,0,4,1) & & \\
(1,4,0,0,1,0) & & & (0,0,4,1,1,0) & & \\
(1,1,1,1,1,1) & \leftrightarrow & \texttt{00101100} & (0,0,2,0,2,2) & \leftrightarrow & \texttt{01101101} \\
(1,0,1,0,0,4) & & & & &
\end{array}$$

of which, three are profile vectors in $\mathbf{p}\mathcal{Q}(8;S)$ (profile vectors of words of length eight).

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How many codewords does Construction I guarantee?

Ehrhart Theory Continues

Define the $(|V|+1+d)\times(|S|+d)$-matrix

$$\mathbf{A}_{\mathrm{GRC}} \triangleq \left(\begin{array}{c|c} \mathbf{A} & \mathbf{0} \\ \hline \mathbf{H} & -p\mathbf{I}_d \end{array}\right).$$

Proposition

If $D(S)$ is strongly connected and $\mathcal{C}(\mathbf{H},\mathbf{0}) \cap \mathrm{Null}_{>0}\mathbf{B}$ is nonempty, then $|\mathcal{C}(\mathbf{H},\mathbf{0}) \cap \mathbf{p}\mathcal{Q}(n;S)|$ is at least the number of lattice points in the interior of the polytope

$$\mathcal{P}_{\mathrm{GRC}} = \left\{\mathbf{u} \in \mathbb{R}^{|S|+d} : \mathbf{A}_{\mathrm{GRC}}\mathbf{u} = (n-\ell+1)\mathbf{b},\ \mathbf{u} \geq \mathbf{0}\right\}.$$

- $\mathrm{Null}_{>0}\mathbf{B}$ denotes the set of vectors in the null space of $\mathbf{B}$ with strictly positive entries.

Theorem

If $D(S)$ is strongly connected and $\mathcal{C}(\mathbf{H},\mathbf{0}) \cap \mathrm{Null}_{>0}\mathbf{B}$ is nonempty, then

$$|\mathcal{C}(\mathbf{H},\mathbf{0}) \cap \mathbf{p}\mathcal{Q}(n;S)| = \Omega'\left(n^{|S|-|V|}\right).$$

- $f(n) = \Omega'(g(n))$ means that for a fixed value of ℓ, there exists an integer λ and a positive constant c so that $f(n) \geq cg(n)$ for sufficiently large n with $\lambda|(n-\ell+1)$.

Systematic Encoding

Objective

Efficient one-to-one mapping

$$\phi : \{0, 1, \ldots, m-1\}^{|S|-|V|-1} \to \mathbf{p}\mathcal{Q}(n; S)$$

such that $\mathbf{v}$ is "embedded" in $\phi(\mathbf{v})$.

For example, 012 encodes systematically into $(3, 1, 0, 2, 1, 1, 2, 2)$, the profile vector of 00000110111100.

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Theorem

Suppose $D(S)$ is Hamiltonian and contains loops. For a suitable choice of m, we can systematically encode $\{0, 1, \ldots, m-1\}^{|S|-|V|-1}$ into $\mathbf{p}\mathcal{Q}(n; S)$.

Systematic Encoding

Construction II

Suppose $D(S)$ is Hamiltonian and contains loops. For a suitable choice of m, if $\mathcal{C}$ is an m-ary $(|S|-|V|-1,d)$-AECC, then $\{\phi(\mathbf{v}) : \mathbf{v} \in \mathcal{C}\}$ is a $(n,d;S)$-ℓ-GRC.

Consider the following code with minimum asymmetric distance 3.

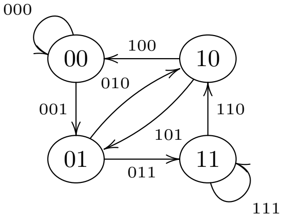
$$\{(0,0,0),(1,4,2),(2,3,4),(3,2,1),(4,1,3)\}.$$

We systematic encode them to 3-gram profile vectors of words of length 8.

$(0,0,0)$	$(0,0,1)$	$(0,1,0)$	$(0,1,1)$	$(1,0,0)$	$(1,0,1)$	$(1,1,0)$	$(1,1,1)$
$(18,$	$0,$	$0,$	$0,$	$0,$	$0,$	$0,$	$0)$
$(1,$	$1,$	$1,$	$4,$	$1,$	$4,$	$4,$	$2)$
$(3,$	$1,$	$2,$	$2,$	$1,$	$3,$	$2,$	$4)$
$(6,$	$2,$	$3,$	$1,$	$2,$	$2,$	$1,$	$1)$
$(0,$	$4,$	$4,$	$1,$	$4,$	$1,$	$1,$	$3)$

This forms a 3-gram reconstruction code of length 20 and distance at least 3.

Corollaries of Main Enumeration Result



Here, $|\overline{\mathcal{Q}}(n;S)| = \frac{n^3}{288} + O(n^2)$.

Theorem (Jacquet, Knessl, Szpankowski, 2012)

Fix q, ℓ and let S be the set of all q-ary strings of length ℓ. Then

$$|\mathcal{E}(n;S| \sim |\mathcal{F}(n;S)| \sim |\overline{\mathcal{Q}}(n;S)| \sim c(S)n^{q^{\ell}-q^{\ell-1}} \text{ where } c(S) \text{ is a constant.}$$

$f \sim g$ means that $\lim_{n\to\infty} f(n)/g(n) = 1$.

Corollary

Suppose $D(S)$ is strongly connected and contains loops. Then

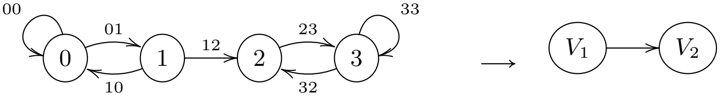
$$|\mathcal{E}(n;S| \sim |\mathcal{F}(n;S)| \sim |\overline{\mathcal{Q}}(n;S)| \sim c(S)n^{|S|-|V|} \text{ where } c(S) \text{ is a constant.}$$

Corollaries of Main Enumeration Result

Results can be extended to enumerate profile vectors of

- all words (not nec. closed) with $D(S)$ strongly connected;
- closed words with $D(S)$ not strongly connected;
- all words with $D(S)$ not strongly connected.

$q = 4,\ \ell = 2,\ S = \{00, 01, 10, 12, 23, 32, 33\}$



Proof Idea when $D(S)$ is not strongly connected: Consider the strongly connected components of $D(S)$ and apply the main enumeration result.

Computing the Leading Coefficient for $\mathcal{Q}(n;S)$

Recall the polytope $\mathcal{P} = \{\mathbf{u} \in \mathbb{R}^{|S|} : \mathbf{A}\mathbf{u} = (n-\ell+1)\mathbf{b},\ \mathbf{u} \geq \mathbf{0}\}$.
The lattice point enumerator

$$L(n-\ell+1) = \#(\mathbb{Z}^n \cap (n-\ell+1)\mathcal{P})$$

gives the number of points in $\mathcal{F}(n;S)$. Hence, the "leading coefficient" for $\mathcal{Q}(n;S)$.
The lattice point enumerator can be computed in polynomial time when the dimension of the polytope is fixed (Barvinok, 1994). However, the dimension of $\mathcal{P}$ is $|S| - |V| \approx q^{\ell}$.

Question

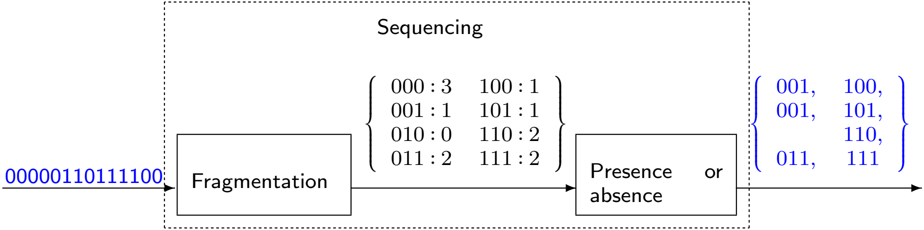
Efficient methods to compute the lattice point enumerator or compute the leading coefficient.

Practical Methods for Profile Counting

Challenge: Counting accurately the number of ℓ-grams.
Instead, we use certain auxiliary information.

- The presence or absence of ℓ-grams
- The relative order of ℓ-grams

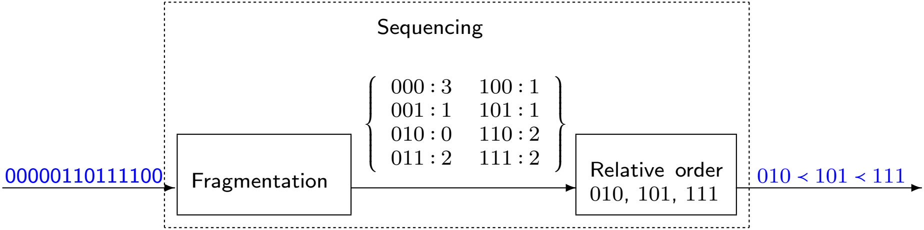
Practical Methods for Profile Counting - Presence or Absence of ℓ-grams



Related combinatorial problems:

- Enumeration of "profile vectors". Tan, Shallit (2013) studied this problem in the context of "factors of words".
- Edge-disjoint path decompositions of de Bruijn graphs. Variety of decompositions surveyed by Heinrich (1993), Bryant and El-Zanati (2007). Cooper and Graham (2004) studied cycle decompositions of de Bruijn graphs.

Practical Methods for Profile Counting - Relative Order of ℓ-grams



Related coding problem:

- Rank modulation codes. Jiang, Mateescu, Schwartz, Bruck (2009) proposed these codes for nonvolatile flash memories.

Growth rate for $\mathcal{Q}(n;S)$ when ℓ grows

When q, ℓ is fixed, $|\mathcal{Q}(n;S)|$ is polynomial in n.
Suppose that q is fixed and ℓ is a function of n, or $\ell = f(n)$.

- For example, when $\ell = n$ and $S = \{0, 1, \ldots, q-1\}^{\ell}$, then $|\mathcal{Q}(n;S)| = q^n$, which has exponential growth in n.

Question

How "small" can ℓ be so as to ensure $|\mathcal{Q}(n;S)|$ has exponential growth in n?

Encoding and Decoding Complexities

Recall that n is the length of codewords.

ℓ-gram distance and code constructions are defined using profile vectors of length $|S| \approx q^{\ell}$.

When $n \le q^{\ell}$, computations based on profile vectors are inefficient.

Ukkonen (1992) showed that (a variant) of the ℓ-gram distance can computed in time $O(qn)$ with space $O(qn)$.

Question

Assume q fixed. Can encoding and decoding by done in time and space polynomial in n?

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